Math In Your World
Solid Geometry: Wood Sculptures by Kosticks

by Ken Fan

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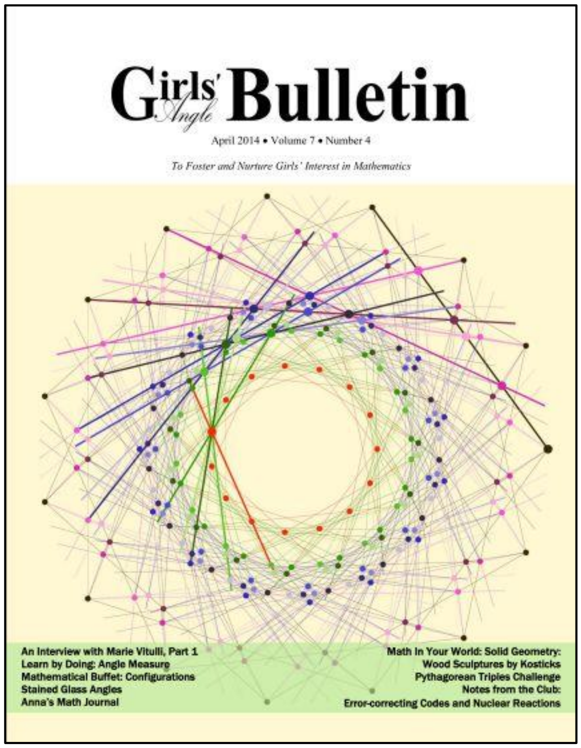

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Solid Geometry: Wood Sculptures by Kosticks

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Art and geometry combine in the minds of John and Jane Kostick. The result is a collection of wondrous wood sculptures that manifest a wealth of interesting mathematical facts.

The best way to get a feel for this mix of math and art is to play with an actual example, and that's just what we'll do. This article contains all you need to make your own paper model of one of the Kosticks' latest creations: the **Quintetra Assembly**.

Inspiration To appreciate the elegance of the Quintetra Assembly, it helps to think about a few more basic shapes with special focus on the directions of their edges. Let's start with a cube, paying particular attention to its 12 edges. Notice that the edges of a cube point in 1 of 3 different directions, just like the axes of a 3D Cartesian coordinate system.

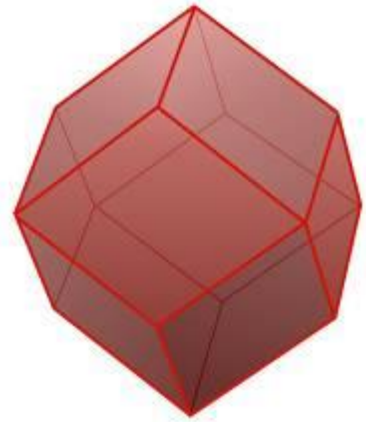
This observation raises the following question: What solids have all of their edges restricted to the same 3 directions as the edges of a cube? Because this restriction is severe, we can get a very good idea of what these shapes look like with a little bit of experimentation. Any brick shape is possible, and so is any solid built by joining bricks together, provided that all of the bricks are consistently oriented to respect the restriction on edge directions. Of these shapes, only the isolated brick will be convex, and of these bricks, the cube is the most symmetric and is the only convex one with edges all of the same length. (A shape is convex if it contains the line segment joining any two of its points. For example, a circular disk is convex, but an annulus is not.)

Let's make a game of this, now using a different set of allowed directions: the 4 directions specified by the **major diagonals** of a cube. The major diagonals are the line segments that connect opposite vertices. What shapes can you find whose edges are each parallel to one of these 4 directions? Note that if we use only 2 of the 4 directions to travel in a circuit by moving in one direction, then the other, then back in the first direction, then back to the starting point in the second direction, we will trace out a parallelogram. Also, keep in mind that if we wish to stay within a plane, we have to restrict ourselves to using just 2 of the 4 directions. Therefore, such solids, if convex, must have faces that are parallelograms. By analyzing the angles between pairs of directions, we find that these parallelograms involve 2 specific angles, namely $\cos^{-1} 1/3 \approx 70.5°$ and its supplement, $\cos^{-1} -1/3 \approx 109.5°$.

Is there an equilateral convex solid whose edges are each parallel to one of the 4 major diagonals of a cube? If such a shape existed, all of its faces would have to be congruent rhombi.

Think about this before reading further.

Continued on page 3



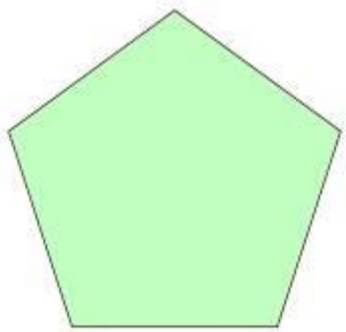
A rhombic dodecahedron.

There is such a solid, and it is called a **rhombic dodecahedron**.

A nonconvex example of a shape whose edges are all parallel to the 4 major diagonals of a cube is the Kosticks' **Tetraxis** puzzle. The name comes from the fact that the edge directions are parallel to the 4 major diagonals of a cube. The video *Tetraxis Geometry* visually explains the geometry of the rhombic dodecahedron and Tetraxis. You can watch it on the Girls' Angle YouTube channel.

> Restrict yourself to the directions defined by the diagonals of the faces of a fixed cube. Find an equilateral convex solid whose edges each run parallel to one of these 6 directions.

A Leap of Imagination We're ready to explain the Quintetra Assembly. Instead of exploring shapes whose edges are all parallel to a fixed set of 3 or 4 directions, as we have done so far, the Kosticks explored shapes whose edges are parallel to a fixed set of 30 directions!



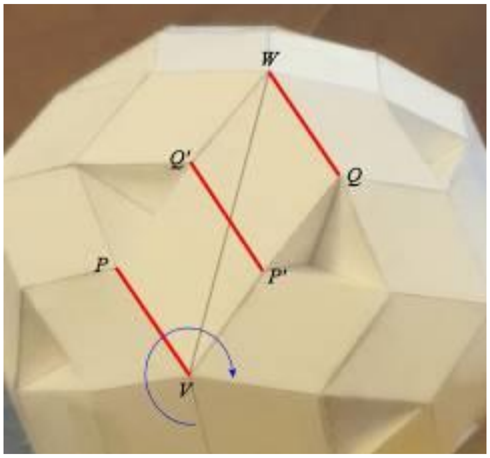
What 30 directions? Start with a regular dodecahedron. A regular dodecahedron is one of the five **Platonic solids**. It has 12 faces that are congruent regular pentagons, with 20 vertices and 30 edges. Three edges emanate from every vertex. To get a good feeling for the shape, build one! If you make 12 copies of the regular pentagon shown at left, you will find that the dodecahedron practically assembles itself because there is little choice for how to put the faces together. You can also turn to page 11 of Volume 3, Number 4 of this *Bulletin* and find the net of a regular dodecahedron that you can print out and fold.

The 20 vertices of the dodecahedron can be grouped into 5 sets of 4 vertices each. In each set, the 4 vertices are the vertices of a regular tetrahedron. If done properly, each of the 5 vertices of any pentagonal face will belong to a different tetrahedron. A tetrahedron has 6 edges, so these 5 tetrahedra collectively have 30 edges. These 30 edges represent the 30 directions to which the Kosticks restricted their explorations.

The Kosticks managed to discern the amazing equilateral convex polyhedron[1] shown at right. By construction, each of its edges runs parallel to one of the 30 directions. The polyhedron consists of 20 equilateral triangular faces and 60 congruent rhombic faces. It has 72 vertices and 150 edges. The centers of the triangular faces form the vertices of a regular dodecahedron, and the rhombi are laid out like a path between the triangles. Most of the vertices are surrounded by 3 rhombi and a triangle, but at 12 of the vertices, 5 rhombi come together to form 5-pointed stars. These 12 special vertices form the vertices of a regular icosahedron.

An equilateral polyhedron with 60 congruent rhombic faces and 20 triangular faces.

[1] According to John Kostick, Zometool is a terrific aid to explore possibilities.



In order to make a model of this polyhedron, the Kosticks had to compute the angles of the rhombic face. One way to find the angles is to determine which of the 30 directions correspond to the adjacent sides of a rhombic face and compute the angle between those 2 directions. I'll sketch another way to find these angles that enables computation of the Cartesian coordinates of all vertices. To follow this approach, you need to be comfortable with trigonometry, vectors, and matrices.

The figure at left shows part of the Quintetra Assembly. Let $\varphi = (1+\sqrt{5})/2$. Vertices V and W are 2 of the 12 vertices where 5 rhombi meet. These 12 vertices form the vertices of an icosahedron. We exploit the fact that the 12 points whose Cartesian coordinates are $(\pm 1, 0, \pm\varphi)$, $(\pm\varphi, \pm 1, 0)$, and $(0, \pm\varphi, \pm 1)$ are the vertices of an icosahedron (where all possible combinations of signs are taken). Without loss of generality, we may assume that $V = (1, 0, \varphi)$ and $W = (-1, 0, \varphi)$.

The 180° rotation about the line that passes through the origin and the midpoint of segment VW interchanges P' and Q'. Therefore, segment $P'Q'$ is parallel to the planes that are perpendicular to the axis of rotation, which include the xy-coordinate plane. That is, P' and Q' have the same z-coordinate. Because $VPQ'P'$ and $WQP'Q'$ are rhombi, we know that PV and WQ are parallel to $P'Q'$. Hence, P, V, Q, and W all have the same z-coordinate, which is φ. Let $P = (x, y, \varphi)$. We seek x and y. By symmetry, we know that $Q = (-x, -y, \varphi)$.

The 72° rotation about the line that passes through the origin and V in the direction indicated by the blue arrow sends P to P'. We use this fact to express the coordinates of P' in terms of the coordinates of P. After some linear algebra, we find

$$P' = \left(\frac{x}{2} + \frac{\varphi}{2} y + \frac{1}{2}, -\frac{\varphi}{2} x + \frac{1}{2\varphi} y + \frac{\varphi}{2}, \frac{1}{2\varphi} x - \frac{y}{2} + \frac{4\varphi+3}{2\varphi+4} \right).$$

Next, we use the fact that $Q'P'$ is parallel to and the same length as PV. This can be expressed by saying that the vector that points from Q' to P' is the same as the vector that points from P to V. When this condition is expressed mathematically and simplified, we arrive at the following system of linear equations in the unknowns x and y:

$$\begin{aligned} 2x + \varphi y &= 0 \\ x - y &= 1 \end{aligned}$$

Solving these for x and y and substituting into our expressions for P and P', we find

$$P = \left(\frac{\varphi}{\varphi+2}, \frac{-2}{\varphi+2}, \varphi \right) \text{ and } P' = \left(\frac{1}{\varphi+2}, \frac{1}{\varphi+2}, \varphi + \frac{2-\varphi}{\varphi+2} \right).$$

From these, we can compute the angle $P'VP$ (for instance, by using the dot product). We find that angle $P'VP = \cos^{-1}(\varphi/4)$, which is approximately 66.14°.

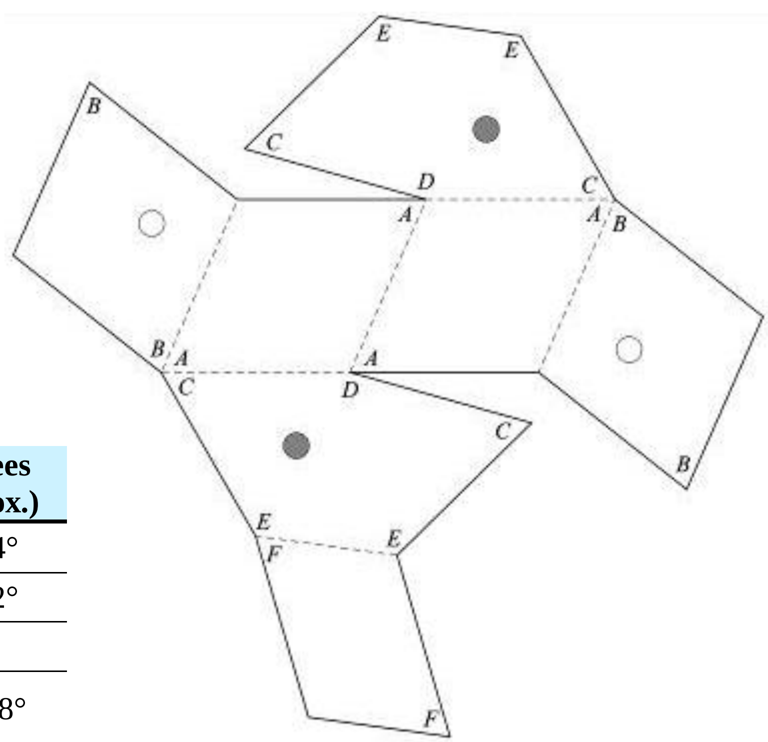
Jane went beyond understanding the surface of the polyhedron. She designed a unique block, called the **Quintetra block**, from which the polyhedron can be built. The Quintetra block consists of 4 rhombic faces, 2 pentagonal faces, and 1 parallelogram face. It takes 30 Quintetra blocks to build the polyhedron.

Photo courtesy of the Kosticks

The Kosticks' Quintetra Assembly.

The lower left image shows the Kosticks' Quintetra block in 3 different types of wood. The image on the right shows the completed model.

Take It To Your World Make 30 copies of the net shown at right. Cut on the solid lines and fold on the dotted lines. Glue or tape the blocks together so that the dark circles connect to the light circles.



Angles The table below gives the measures of angles in the net. If an angle is unmarked, it is part of a parallelogram with a marked angle.

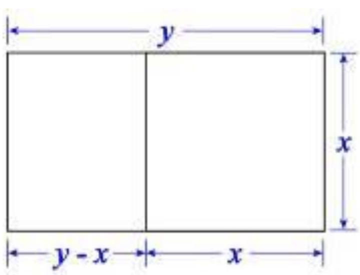
Angle	Exact Measure	Degrees (approx.)
A	$\cos^{-1}(\varphi/4)$	66.14°
B	$\cos^{-1}(1/4)$	75.52°
C	60°	60°
D	$\cos^{-1}((1 - 3\varphi)/4)$	164.48°
E	$210° - D/2$	127.76°
F	$\tan^{-1}\sqrt{5}$	65.91°

More Amazing Facts We urge you to build the model. It will help you to follow the remainder of this article.

There are 20 dimples in the Quintetra Assembly. Recall that all edges of the polyhedron run parallel to the directions determined by the 30 edges of the 5 regular tetrahedra obtained from the 20 vertices of a regular dodecahedron. If you extend a tetrahedron's edges beyond its vertices, you will form "inverted" tetrahedra, and if you extend the edges of these 5 direction-determining tetrahedra, you get the dimples of the Quintetra Assembly!

If you pick out one of the triangular faces T, you can walk from it to 3 other triangular faces on a path of rhombi that each contain edges parallel to the sides of T. These 3 triangular faces, brought together with T (without changing their orientation), will make a regular tetrahedron.

To explain our last observation, we must describe the **rhombic triacontahedron**. A rhombic triacontahedron is a solid with 30 congruent **golden rhombi** for faces. A golden rhombus is formed by connecting the midpoints of the sides of a **golden rectangle**. A golden rectangle is a rectangle with a unique property: if you chop off the largest square possible from one side, the leftover piece will be a rectangle with the same proportions as the original.



Let's walk backwards through these definitions in more detail. Shown at left is an x by y rectangle. A vertical line is drawn inside to mark the left edge of an x by x square. The defining property of a golden rectangle is that the x by $y - x$ rectangle that remains after removing the x by x square is similar to the x by y rectangle. That is, $x : y - x = y : x$. Cross-multiplying, we get $x^2 = y(y - x)$, or $(y/x)^2 - (y/x) - 1 = 0$. This is a quadratic equation in y/x. Since the ratio is positive, we find that $y/x = \varphi$.

Now that we know the exact proportions of a golden rectangle, we can illustrate a fine example, shown at right. To get the golden rhombus, we connect the midpoints of the 4 sides as shown below left. We cut along the lines. The result is the golden rhombus shown at right. Show that the smaller angle in the golden rhombus has a measure of $\tan^{-1} 2$, which is about 63.435°.

To build a rhombic triacontahedron, make 30 of these golden rhombi, all the same size. Join them edge-to-edge to build a 3-dimensional solid. As you join the faces, like angles should meet like angles: 5 golden rhombi meet at each acute-angled vertex, and 3 meet at each obtuse-angled vertex.

What was the point of describing the rhombic triacontahedron? The interior of the Quintetra Assembly is empty. Amazingly, this space will snugly receive a rhombic triacontahedron! So snug is the fit that each of the rhombic triacontahedron's faces will be flush with the inside face of a Quintetra block. For a challenge, prove this fact and find the exact size relationship needed for a snug fit.

Photo courtesy of Jane Kostick

A rhombic triacontahedron built by Jane Kostick.

For more examples of the Kosticks' work and to learn more about the fascinating properties of the Quintetra Assembly, visit their website at www.kosticks.com.