Space Filling with Polar Zonohedra

A study of cellular packing in n-directional coordinate systems

by John M. Kostick

This project has its origins in space frames and other symmetry-based structures that I was making decades ago, using wood, wire, glass, plastic, paper etc. More recently, with the use of CAD drawings and [Zometool]() models, I have been occasionally exploring configurations in this particular family of spatially reiterative patterns. Over the last couple of years, since starting to use [vZome](), many of these patterns are now represented in the models shown here. Enduring thanks to Scott Vorthmann and David Hall for creating vZome and for adapting it to my uses.

John Kostick, June 2020

"A **zonohedron** (by one restrictive definition) is a convex polyhedron all of whose faces are parallelograms." (See George Hart's [website]() for reference and in-depth discussion.)

There are several types of zonohedra, including cubes (more broadly hexahedra) with three directions of edges, and rhombic dodecahedra with four directions of edges. Both of these are so-called uniary space fillers, which means they can pack together to occupy space with no gaps, and only the one kind of cell.

Here we are looking at zonohedra with more than four directions of edges, which mostly have just a single axis of symmetry. These polar zonohedra can be elongated along an axis, like a football (prolate), or squashed down to more like a lozenge (oblate).

A 5-directional zonohedron is a rhombic icosahedron. There is a canonical version of this polyhedron which has 20 faces of just one kind, the golden rhombus, with ArcTan 1/2 = 63.43…° between edges. This is essentially a triacontahedron, (30 faces) which is also a zonohedron, with one zone reduced to zero, or eliminated. Many other types of regular rhombic icosahedra having two kinds of rhombic faces can be constructed with other 5-fold symmetry angular systems.

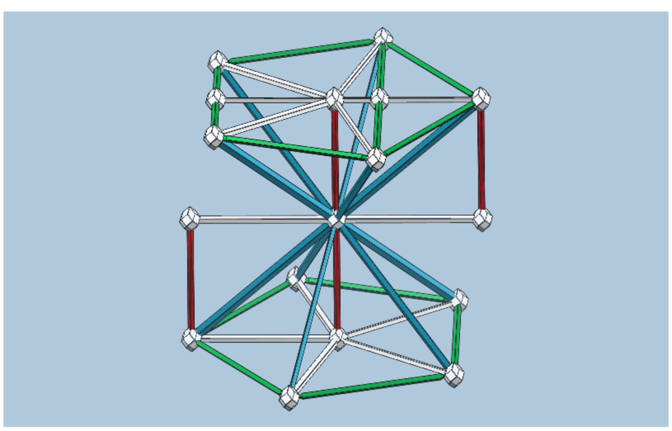

Figure 1

Consider Figure 1. Given a pentagonal pyramid with base edges e (in green) and sloping edges E (in blue), the height H (in red) can be set so that nonadjacent E edges are perpendicular.

We call this the normalized 5-directional coordinate system, because the nonadjacent E edges are normal to each other. It is also referred to as the √ϕ field. There is just one axis of 5-fold rotational symmetry, and so H can be set to make the framework conform to any 5-fold system, such as can be built with blue, yellow, green as well as red Zometool struts. These can range from extremely elongated (prolate) to extremely flattened (oblate). In this configuration, the angle between adjacent E edges is arccos(1/ϕ) = 51.827...° An unusual property of this version is that the dihedral angles of the polyhedron are of the same set as the face angles.

Treating the apex of the pyramid as an origin point, segments of unit length E can be extended from endpoints always parallel to an original E, generating rhombi and squares, and polyhedra composed of them. The major polyhedron is the rhombic icosahedron, RI.

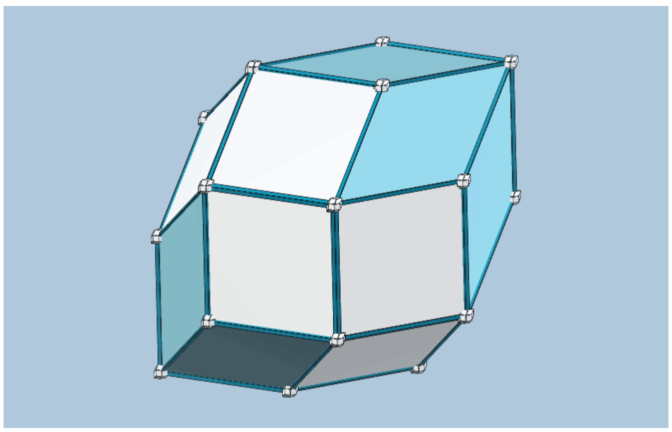

Figure 2

Figure 2 shows an RI defined thus, with ten square faces and ten 51.827...° rhombic faces.

Figure 3 shows a dissection of an RI, where the major cell with one zone reduced to zero is an eccentric rhombic dodecahedron, shown in purple. This cell can be further reduced by one zone in two different ways, yielding two kinds of hexahedron, shown in blue and in red. This is generally the case with zonohedra: A major cell with n edge directions reduces by one zone to a cell with n-1 zones, etc. (For polar zonohedra, the number of kinds of hexahedron that are defined by this process is n-3.) For a 3-D view of dissection of an RI, see the model link 4.

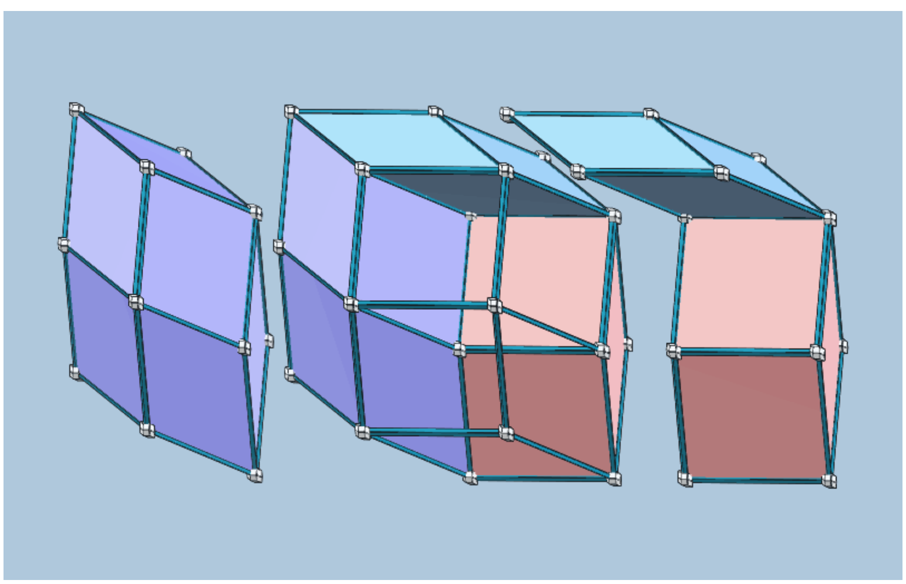

Figure 3

In general, regular polar zonohedra with n zones can be composed of unit edge lengths distributed symmetrically around an axis so that the cell has n-fold rotational symmetry and n planes of reflection intersecting the axis. For cells where n is odd, the array typically has a direction with respect to the polar axis, (such as up/down or heads/tails, like an arrow.) Where n is even, the symmetry is typically dipolar, where the "up" and "down" directions of the array can be equivalent. (Zonohedra can have various lengths, each zone is effectively an independent variable. Cells can be modulated, just as a cube can be modulated into a brick shape, where length, width, and height are each different. Whole n-directional arrays can in fact be composed where any number of lengths are employed in different zones in separate regions of the array.)

Polar zonohedra can be connected one to another at shared polar vertices, or nodes, like strings of beads. They can be joined thus in a parallel manner, so that the edges of all cells in a string are aligned with the original coordinate directions.

Cells can also be connected by a shared face, by a shared edge, or by other shared vertices, also maintaining a parallel relationship with the original coordinates. One can compose configurations of connected cells that form arrays that are in some ways periodic and/or that conserve the symmetry of the original cell. This leads to a concept of n strings of "beads" surrounding the original string, and courses of strings in multiples of n surrounding them.

A working hypothesis is that an array, or arrays of polar zonohedra can be built that fills space using only line segments with directions of the given coordinate system. The array can have some or all of the symmetry characteristics of the single major cell. All the spaces between major cells are occupied with minor, or reduced cells (subcells) in the system, i.e. all edges are parallel to one of the original directions.

In the model link 1 two RI cells are connected at a common point. The rest of the spaces surrounding that point are filled with ten hexahedral wedge shaped subcells. It can be seen that there are "nests" where RI cells fit into the array, each sharing a face with one from an original RI cell. Here there is a choice: One can fit fit five RI cells in, but this precludes fitting RI cells into the other five "nests". These can only accommodate cells that are reduced to eccentric rhombic dodecahedra, as in the dissection. This is more apparent looking at the model link 2. This choice gives the whole array a directional character in relation to the central axis, there is a "top" and a "bottom" to the array.

These arrays can be understood as spatial patterns akin to tiling patterns in planes.
Model link 30 illustrates a series of planes that intersect a 5-directional array, through select vertices. The axis of the two RI cells is normal to all these planes. The polygons in these planes represent the polyhedra in the array, they are actually sections through polyhedra as they occur in the array. It is apparent that these tiling patterns can all continue, maintaining rotational symmetry and lines of reflection throughout.
In building out this kind of array, it is a strategic consideration to always use the largest cell that fits into the "nests." This allows maintaining the symmetry of the primary unit in

the array. It is possible to dissect a rhombic dodecahedron, for example, into four hexahedra, but doing so reduces the overall symmetry. Using a hexahedron where a dodecahedral cell could be used typically reduces the symmetry. In some cases this can be done intentionally, making for a left-handed or right-handed (chiral) array, which has rotational symmetry but no planes of reflection.

The model link 3 shows an extended packing of RI cells without much explicit filling in of the spaces between. It can be seen that this whole array has a “head” and a “tail.”

Model link 5 is an extended array showing one of the five planes of reflection symmetry. In this model, the directional character of the array is also very evident.

Model link 7 shows how a dipolar 5-directional array can be composed, by building two arrays of opposite direction that converge around a common origin.

Model link 6 shows a different pattern that can be composed, building outward from an RI using only subcells of the system, to arrive at an RI of double the scale. This is self-similarity. This larger RI can be built outward from in the same way, thus arriving at an RI of 4x the scale. This process can be done recursively to fill space. This is a chiral array, it can be built right handed or left handed with respect to the direction of the axis.

Six-directional polar zonohedra can be composed in more than one way. The major type of cell is a polar triacontahedron, (30 facets) which has three kinds of facets. Model link 10 shows a major cell and all the kinds of subcells that can be derived from it by reducing zones. These can be seen as dissection components. Depending on which zones are reduced to zero, the subcells are: one type of eccentric icosahedron, two types of dodecahedron, and three types of hexahedron.

Model link 9 shows the beginning of a space filling array using just the triacontahedron cells, one type of dodecahedron, and one type of hexahedron, which are cubes in two orientations. It can be seen that the 6-directional coordinate system consists of two 3-directional (Cartesian) systems that share an origin point, but which are rotated 60° from each other about the axis of symmetry. This array can have six-fold rotational symmetry, six reflection planes intersecting the axis, and can also be composed so that it is dipolar, with a plane of reflection through the origin, the axis being normal to it. The major cells are packed with shared faces in hexagonal arrays, so the array can be periodic in directions to which the axis is normal. The arrangement is six strings of beads surrounding and in contact with the original string. Surrounding each “node,” or shared polar vertex, are six dodecahedral cells The remaining spaces are all cubes, in the two orientations. Model link 11 shows an extended array.

Model link 8 shows another version of a 6-directional array which also has two 3-axis (Cartesian) coordinate systems, which are rotated from each other by a different amount (~ 75.52° or 44.48°), so that the triacontahedron has four types of facets, and the overall symmetry is reduced to 3-fold. (The orientation of the two cubic systems in this version is taken from how they occur as two of the five cubic systems inherent in icosahedral symmetry.)

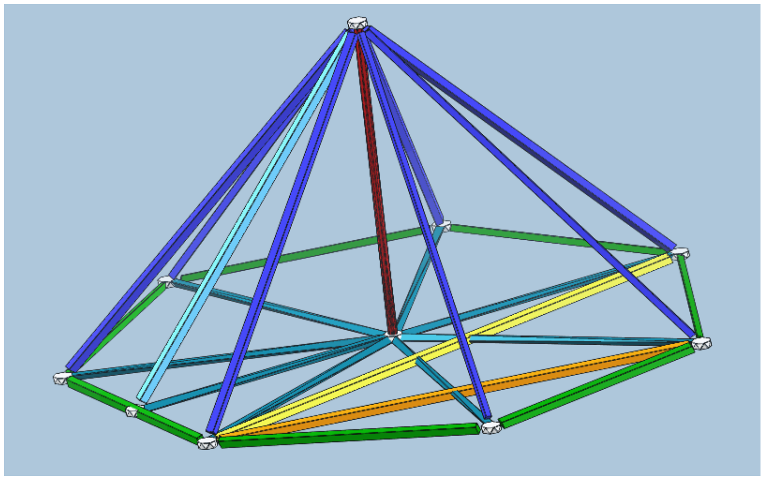

Figure 4

Figure 4 shows a heptagonal based pyramid, which is how the 7-directional coordinate system is defined, and thus how the 7-fold polar zonohedron is generated. As with the 5-directional system, the height of the pyramid can be chosen to set the angles between sloping edges. The edges that are farthest apart, as shown connected by a yellow chord in the base plane, can be set to be perpendicular, or normal. This generates a football-shaped, 42-faced zonohedron with 14 square faces and 14 each of two other types of rhombi. (This is typical of how n-directional coordinate systems that can generate n-fold polar zonohedra are defined. The apex of a pyramid with a polygon base is an origin point.)

Model link 12 shows such a 42-hedron with a zone indicated by purple edges and another indicated by green edges. Reduction of the purple zone to zero leaves an eccentric triacontahedron. There are four types of hexahedron to be found in this system.

Model link 14 shows how these cells are arranged, connected at nodes, with a ring of seven cells surrounding the original string, or chain. With the 5-fold arrays, the "diameter" of a single cell is greater than the "diameter" of the opening within a ring of five cells, so the members of the ring must nest into the array at a different position along the axis than the original cell, "above" or "below" it. With the 6-directional arrays, the "diameter" of the cells is the same as the "diameter" of the opening within the ring of six cells, so the cells are in contact at the same position along the axis, which allows for the hexagonal character of the array. With the 7-fold arrays, the "diameter" of the cells is less than the "diameter" of the opening within the ring of seven cells, so there is no direct contact. This necessitates filling the space with a connecting bridgework of subcells.

In this case it can be seen that there are seven hexahedral cells that connect the original cell to the surrounding ring. The position of the center of the ring along the axis is not the same as the position of the center of the original cell, but is offset, as is the 5-fold. The remaining space can be fully occupied with a selection of subcells only using unit lengths of the system. (This is the case with all arrays where n > 6: The diameter within the rings is always larger than a cell.)

Model link 15 shows a more extended array, with a course of 14 cells surrounding the ring of 7, connected in the same way, with the same offset. This clearly displays the directional character of the array.

Model link 16 shows a buildout from a single cell to a self-similar double scale cell.

Model link 13 shows 7-fold array based on a different set of angles, a more squashed down, or oblate 42-hedron. All other features of the array are the same.

Model link 29 shows one of the alternate versions of 8-directional array.

Figure 5 is a 2-D representation of an extension of this array, as an octagonal tiling pattern.

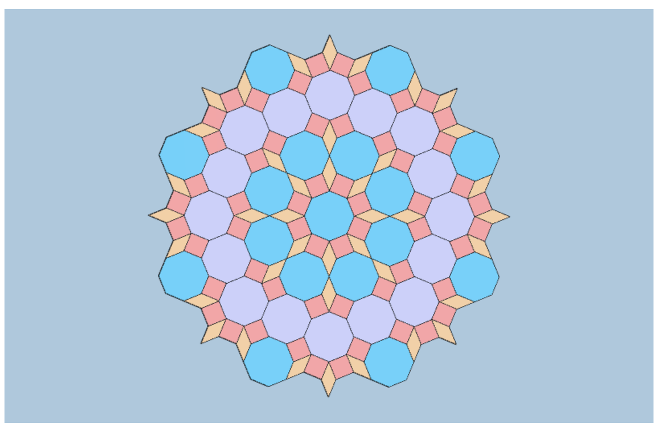

Figure 5

Most of the remaining model links are more or less self-explanatory, each having some descriptive text. Model links 26 and 27 show two alternate ways to build a 12-fold ring, and thus the bridgeworks of subcells. Model link 21 shows how four separate but parallel zones continue throughout different parts of an array. Model link 28 likewise shows two parallel zones that continue throughout different parts of another array.

Altogether, the working hypothesis appears to hold true for these models. While this study does not purport to be a proof, or even a particularly thorough explanation of how these arrays are discovered, it is intended to serve as a demonstration of an interesting class of space-filling patterns.

Links to 3-D Models

1) Rhombic Icosahedron Space Filling — https://skfb.ly/6QCYF

2) Space-Filling Array in √ϕ Field — https://skfb.ly/6QDnL

3) Rhombic Icosa packing — https://skfb.ly/6PqzS

4) Rhombic Icosa Dissection — https://skfb.ly/6KVUX

5) 5-Directional Field — https://skfb.ly/6QDnB

6) 5-Directional Recursive Space Filling — https://skfb.ly/6QDnD

7) Dipolar Space Filling in 5 Directions — https://skfb.ly/6P9NY

8) Space Filling with 2 Cubic Systems — https://skfb.ly/6QCYV

9) Space-Filling Array with Polar Triacontahedra — https://skfb.ly/6QCZq

10) Polar Triacon Dissection — https://skfb.ly/6KWOO

11) Polar 6-Directional Space-filling Array — https://skfb.ly/6QDnH

12) 7-Zonohedron Dissection — https://skfb.ly/6LnSR

13) Seven Space — https://skfb.ly/6OuAp

14) 7-Directional Space Filling — https://skfb.ly/6QCZo

15) 7-Directional Space-Filling Array — https://skfb.ly/6QCYZ

16) 7-Directional Self-Similar Array — https://skfb.ly/6QDnI

17) 8-Directional Space-Filling Array — https://skfb.ly/6QCZs

18) 8-Directional Space-Filling Array v.2 Extended — https://skfb.ly/6QCZW

19) 8-Directional Array 3 — https://skfb.ly/6QCZQ

20) 8-Directional Self-Similar Array — https://skfb.ly/6QCZY

21) Zones in an 8-Directional Array — https://skfb.ly/6QDnv

22) 9-Directional Space-Filling Array — https://skfb.ly/6QCZv

23) 9-Directional Self-Similar Space Filling	https://skfb.ly/6QDnA
24) 10-Directional Array v.1	https://skfb.ly/6QCZN
25) 10-Directional Space-Filling Array	https://skfb.ly/6QCZz
26) 12-Directional Array v.1	https://skfb.ly/6QJzW
27) 12-Directional Array v.2	https://skfb.ly/6QCZD
28) Parallel Paths in 14-Directional Field	https://skfb.ly/6QDzH
29) 8-Directional Space-Filling Array v.2	https://skfb.ly/6QN6p
30) 5-Fold Planes	https://skfb.ly/6QNSy